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ON CONTROLLED ROTATION OF AN ELASTIC ROD*

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Plane rotational motions of an elastic rod loaded by a perfectly rigid body and acted upon by a controlling moment of forces, are considered. A system of integro-differential equations with initial and boundary conditions is obtained. The problems of control are studied, which carried the system from some initial state to a given angular state with damping of elastic oscillations or to a state when the system rotates as a whole with fixed angular velocity. These formulations appear in the course of considering a whole series of practical problems of controlling the systems with elastic constraints such as robots and manipulators, weight lifting machines, etc. The asymptotic methods are used to obtain the solution of the control problems stated, close to the two limiting cases: 1) the case of weightless rod (quasistatic approximation) and 2) the case of high flexural rigidity. The problems of dynamics and control of oscillating systems with distributed parameters were studied in /1—11/ et al.

1. Equations of controlled motion of an elastic rod. We consider a mechanical system representing an elastic rod of variable cross section able to rotate in a certain plane (Fig.1). One end of the rod is fixed (point O) in the inertial $OX'Y'Z$ space, and a perfectly rigid body G is attached to the other end. The linear dimensions of G are assumed to be small compared with the rod length. The axis of rotation OZ passes through the point O in the direction perpendicular to the plane of motion, and the controlling force moment is applied relative to this axis. To describe the motion we introduce the $OXYZ$ -coordinate system rotating in the inertial space, with the common OZ axis. The OX axis is projected along the direction tangent to the neutral line of the rod at the point O. We assume that the motion of the model is described within the framework of the linear theory of thin, rectilinear, inextensible rods /1,2,9/. The elastic displacements are assumed small and perpendicular to the OX axis coinciding with the neutral line of the undeformed rod.

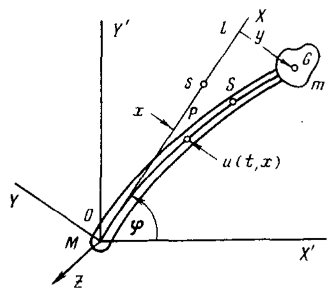

Fig.1

We introduce the following notation (some of it is shown in Fig.1): x and s are the abscissas of two points P and S respectively, in the moving system OXY, $0 \leqslant x \leqslant s \leqslant l$, l is the rod length and is constant, $\rho(x)$ is the linear density, $\rho_1 \leqslant \rho \leqslant \rho_2$, $\rho_{1,2} > 0$, E is the Young's modulus, $I(x)$ is the moment of inertia of the transverse cross section about the axis perpendicular to the plane of flexure $I_1 \leqslant I \leqslant I_2$, $I_{1,2} > 0$ /2/, m is the mass of the body G situated at the rod end (at the point $x = l$), M is the concentrated moment of control forces about the axis of rotation OZ, φ is the angle between the OX axis tangent to the elastic axis of the rod at the point O and the OX' axis, $u(t, x)$ is the displacement of the point on the elastic axis of the rod with coordinate x at the instant t. Depending on the formulation of the control problem the variables M and φ are either given or sought functions of t, and $u(t, x)$ is a sought function.

Let us derive the equations of motion of the mechanical system in question. We denote by $\mathbf{x}$ the radius vector of the point lying on the OX axis with abscissa x, and by $\mathbf{u}(t, x)$ the displacement vector of the point x lying on the neutral line of the rod, $\boldsymbol{\omega}, \boldsymbol{\gamma}$ are, respectively, the angular velocity and acceleration of the $OXYZ$ coordinate system relative to the inertial $OX'Y'Z$ frame of reference. We have the following coordinate representations for the above vectors in the $OXYZ$ system:

$$\mathbf{x} = \begin{Vmatrix} x \\ 0 \\ 0 \end{Vmatrix}, \quad \mathbf{u} = \begin{Vmatrix} 0 \\ u \\ 0 \end{Vmatrix}, \quad \boldsymbol{\omega} = \begin{Vmatrix} 0 \\ 0 \\ \varphi^{\cdot} \end{Vmatrix}, \quad \boldsymbol{\gamma} = \begin{Vmatrix} 0 \\ 0 \\ \varphi^{\cdot\cdot} \end{Vmatrix} \tag{1.1}$$

*Prikl.Matem.Mekhan.,46,No.4,pp.587-595,1982 where a dot denotes differentiation with respect to time t.

Let P be any point of the rod; then $\mathbf{r}(t, x) = \mathbf{x} + \mathbf{u}(t, x)$ is the radius vector of this point at the time t. Let us compute the total (relative to $OX'Y'Z$) velocity $\mathbf{v} = \mathbf{r}^{\cdot}$ and acceleration $\mathbf{w} = \mathbf{r}^{\cdot\cdot}$ according to the known kinematic principles /1/

$$\mathbf{v}(t, x) = \mathbf{u}_t + \boldsymbol{\omega} \times \mathbf{r}, \quad \mathbf{r}_t = \mathbf{u}_t \qquad \mathbf{w}(t, x) = \mathbf{u}_{tt} + \boldsymbol{\gamma} \times \mathbf{r} + 2\boldsymbol{\omega} \times \mathbf{u}_t - \omega^2 \mathbf{r}, \quad \mathbf{r}_{tt} = \mathbf{u}_{tt} \tag{1.2}$$

The subscript t denotes a partial derivative with respect to t calculated in the rotating $OXYZ$ coordinate system. We derive the equation of relative motion of the points on the rod by taking any point P on its neutral line with coordinate x, the radius vector of which is $\mathbf{r}(t, x) = \mathbf{x} + \mathbf{u}(t, x)$, see (1.1). Applying an orthogonal cut we divide the rod into two subsystems, OP and PG. Let us find the principal moment $\mathbf{N}$ of the D'Alembert's forces of inertia acting on the elements of the subsystem PG relative to the point P. Integrating over all elements of the rod for $x \leqslant s \leqslant l$, we obtain

$$\mathbf{N}(t, x) = -\int_x^l \rho(s)\,[\mathbf{r}(t, s) - \mathbf{r}(t, x)] \times \mathbf{w}(t, s)\, ds - m\,[\mathbf{r}(t, l) - \mathbf{r}(t, x)] \times \mathbf{w}(t, l) \tag{1.3}$$

According to (1.1) — (1.3) the vector $\mathbf{N}$ has a nonzero projection on the OZ axis only. In the linear approximation with respect to u and u_t, we obtain

$$\begin{aligned} N(t, x) = &-\int_x^l \rho(s)\,\{(s - x)\,u_{tt}(t, s) + \varphi^{\cdot\cdot}(s - x)\,s\,ds - \\ &\varphi^{\cdot 2}[u(t, x)\,s - u(t, s)\,x]\}\,ds - \\ &m\,\{(l - x)\,u_{tt}(t, l) + \varphi^{\cdot\cdot}(l - x)\,l + \varphi^{\cdot 2}[x u(t, l) - l u(t, x)]\} \end{aligned} \tag{1.4}$$

Let us now equate the principal moment N of the D'Alembert's inertia forces (1.4) with the moment of "external" elastic forces acting at the cross-section P on the subsystem PG from the direction of the subsystem OP and taken with the opposite sign. The moment of elastic forces is equal, in accordance with the adopted theory of slight flexure of thin rods, to EIu_{xx} (see /2/, the index x denotes the corresponding partial derivative). As a result we obtain the following integrodifferential relation for all t and $0 \leqslant x \leqslant l$:

$$N(t, x) = EI(x)\,u_{xx}(t, x) \tag{1.5}$$

Differentiating the identity (1.5) twice with respect to x, which we assume allowed, we obtain the required partial differential equation describing small elastic deflections of the rod $u(t, x)$, relative to the moving $OXYZ$ coordinate system, in the form

$$\rho(x)\,u_{tt} + E\,[I(x)\,u_{xx}]_{xx} = \varphi^{\cdot 2} u_{xx} \int_x^l \rho(s)\,s\,ds - \varphi^{\cdot\cdot}\rho(x)\,x - \varphi^{\cdot 2}(u_x x - u)\,\rho(x) + ml\varphi^{\cdot 2} u_{xx} \tag{1.6}$$

In addition, the function $u(t, x)$ must satisfy, at every instant of time, the boundary conditions at $x = 0$ and $x = l$

$$\begin{aligned} &u(t, 0) = u_x(t, 0) = u_{xx}(t, l) = 0 \\ &E\,[I(x)u_{xx}(t, x)]_x\,|_{x=l} = m\,\{u_{tt}(t, l) + \varphi^{\cdot\cdot} l + \varphi^{\cdot 2}\,[l u_x(t, l) - \\ &\quad u(t, l)]\} \end{aligned} \tag{1.7}$$

The first two conditions (for $x = 0$) are obvious; they follow from the properties of the $OXYZ$ system and have geometrical character. The third and fourth condition (for $x = l$) are dynamic and follow directly from (1.4), (1.5) and the derivative of (1.5) in x. In the case when the linear dimensions of the body G are appreciable (commensurate with the rod length l), the expressions given above become more bulky, and those of the type (1.3) — (1.7) more complicated. The problem however essentially remains the same.

To extend the equations of motion of the elastic system for a given external force moment $M(t)$, we must supplement the equation (1.6) and boundary conditions (1.7) with the equation describing the change of the angular momentum of the whole system relative to the OZ axis and the corresponding boundary conditions. Following the previous arguments, we obtain

$$\int_0^l \rho(x)\,[\varphi^{\cdot\cdot} x^2 + x u_{tt}(t, x)]\,dx + ml\,[l\varphi^{\cdot\cdot} + u_{tt}(t, l)] = M(t) \tag{1.8}$$

The left-hand side of the integrodifferential equation (1.8) represents the total derivative with respect to time t, of the angular momentum of the system relative to the axis of rotation OZ (in the linear approximation in u and u_t). The angular momentum is calculated from (1.1) and (1.2) in the same manner as (1.4).

To determine the motion of the system uniquely, we specify the initial configuration and velocity of the points on the neutral line of the rod, as well as the initial values of φ and $\varphi^{\cdot}$

$$u(0, x) = f(x), \quad u_t(0, x) = g(x), \quad 0 \leqslant x \leqslant l \tag{1.9}$$
$$(f(0) = f'(0) = 0), \quad \varphi(0) = \varphi_0, \quad \varphi^{\cdot}(0) = \varphi_0^{\cdot}$$

Thus we have obtained a system of integrodifferential equations in partial derivatives (1.6), (1.8). When the boundary (1.7) and initial (1.9) conditions are taken into account and the external moment $M(t)$ of the control forces given, the system defines uniquely the motion of the mechanical system in question, in the linear approximation with respect to the elastic deflection. We note that the quantity u is assumed small (linear theory). The quantities $\varphi^{\cdot\cdot}$, $\varphi^{\cdot}$, M may be sufficiently large (of the order of unity) in some problems of dynamics and control. Therefore the control system in question is, generally speaking, essentially nonlinear.

Next we pose, for the system (1.6) — (1.9), the following problem of rotation of a loaded elastic rod with damping of the relative oscillations. To find the admissible control $M \in K$ such that the following relations held for all $0 \leqslant x \leqslant l$:

$$u(T, x) = u_t(T, x) \equiv 0, \quad \varphi(T) = \varphi_*, \quad \varphi^{\cdot}(T) = 0 \tag{1.10}$$

Here K denotes a specified fixed set of admissible values of the control, and T is the time determined in the course of solving the problem (1.6) — (1.10) from certain additional demands (of optimality, etc.). We note that the value of T must be sufficiently large. This is connected with the fact that the velocity of wave propagation in an elastic rod is finite /5/. Moreover, the time of completion T of the control process must be large since the control moment M is assumed small for a rod of finite rigidity, the assumption demanded by the linear theory of elasticity. Since the value of T at the same time may be large, the solution of the control problem must involve a strict assessment of the retained and neglected terms. From (1.6) — (1.8) it follows that the system will remain at rest (1.10) for $t > T$ if we put $M \equiv 0$ or if we "clamp" the rod at an angle φ_*. In the same manner we pose the problem of bringing the system to the state of uniform rotation, as a whole, with a given velocity $\varphi_*^{\cdot}$ and damping of the relating oscillations

$$u(T, x) = u_t(T, x) \equiv 0, \quad \varphi^{\cdot}(T) = \varphi_*^{\cdot} \tag{1.11}$$

It is clear that the system (1.6) — (1.8) admits a solution of the form (1.11).

2. Approximate approach to investigation of control problems. It is not possible to construct an analytic solution to the mixed boundary value problem and the Cauchy problem (1.6) — (1.9) for the given function $M(t)$, since the variables cannot be separated even when the parameters ρ and I are constant. We note that the problem of motion of a rod under load can be studied using an inverse method. First we construct for the given function $\gamma(t) = \varphi^{\cdot\cdot}(t)$ a solution of the boundary value problem (1.6), (1.7), (1.9) $u(t, x)$, and then use the formula (1.8) to compute the value of the control forces moment $M(t)$, $t \in [0, T]$ required for the motion in question. In this approach the function $\gamma(t)$ is regarded as a control chosen from the targets of the motion (1.10) or (1.11). The controlled system (1.6) — (1.11) can be investigated in approximate manner provided that certain additional assumptions are fulfilled. The analysis and solution of the problem is facilitated by using dimensionless variables characterising the ratios of certain dimensional physical quantities. These dimensionless variables can be introduced by various methods, and we shall consider two of these methods.

1°. We introduce new dimensionless variables as follows:

$$t' = \nu t, \quad u' = u / l, \quad x' = x / l, \quad I' = I / I_0, \quad \rho' = \rho / \rho_0 \tag{2.1}$$

Here ν is a characteristic constant with dimension of frequency, I_0 and ρ_0 are the characteristic parameters of the problem with dimension of the moment of inertia and linear density respectively. As I_0 and ρ_0 we can use, for example, e.g., the values of I and ρ averaged over the interval $0 \leqslant x \leqslant l$. The quantities I' and ρ' in (2.1) are of the order of unity by virtue of Sect.1. The choice of the parameter ν is governed by the specific features of the problem.

Let us choose as ν the quantity $\nu = (EI_0 / ml^3)^{1/2}$ characterizing the frequency of the quasistatic oscillations of the system. Then, using the new variables (2.1) we can describe the motion by equations with the boundary, initial and final conditions (1.6) — (1.11) where the following substitutions should be made:

$$l \to 1, \ m \to 1, \ f \to f' = f / l, \ g = g' = g / l\nu \tag{2.2}$$
$$\varphi^{\cdot} \to \varphi^{\cdot\cdot} = \varphi^{\cdot} / \nu, \quad M \to M' = M / ml^2\nu^2, \quad \rho \to \varepsilon\rho'$$
$$\varepsilon = \rho_0 l / m$$

In what follows we shall omit the primes for convenience. The representiation (2.2) is convenient for the study of the problems of control in, so called, quasistatic approximation when the mass of the rod is much smaller than the mass of the body G., in this case the problem has a small parameter $\varepsilon \ll 1$.

We note that the limiting case of $\varepsilon = 0$ (weightless rod) is also extremely interesting, and was the subject of a number of theoretical and applied investigations /1,3,9/. A very interesting case is that of rotation at thigh angular velocities ($\varphi^{\cdot} > \nu$) leading to considerable increase in the bending strength of the elastic rod /3/, i.e.to increase in the effective rigidity.

Let us therefore consider the problem of control in the quasistatic approximation, in which we take into account only the motion of the mass G, and the weightless rod is in quasistatic equilibrium at any moment of time $t > 0$. Putting $\varepsilon = 0$, we obtain the following equations and boundary conditions:

$$[I(x)\,u_{xx}]_{xx} = \varphi^{\cdot 2} u_{xx},\quad \varphi^{\cdot\cdot} + u_{tt}(t,\ 1) = M(t) \tag{2.3}$$

$$u(t,\ 0) = u_x(t,\ 0) = u_{xx}(t,\ 1) = 0 \tag{2.4}$$

$$[I(x)\,u_{xx}]_x|_{x=1} = u_{tt}(t,\ 1) + \varphi^{\cdot\cdot} + \varphi^{\cdot 2}[u_x(t,\ 1) - u(t,\ 1)]$$

The conditions in the beginning and at the end of the control process for the problem of twisting and rotation will, respectively, become

$$u(0,\ 1) = f(1),\ u_t(0,\ 1) = g(1),\ \varphi(0) = \varphi_0,\ \varphi^{\cdot}(0) = \varphi_0^{\cdot} \tag{2.5}$$

$$u(T,\ 1) = u_t(T,\ 1) = 0$$

$$\varphi(T) = \varphi_*,\ \varphi^{\cdot}(T) = 0 \tag{2.6}$$

$$\varphi^{\cdot}(T) = \varphi_*^{\cdot} \tag{2.7}$$

We note that the initial distribution of the points of the weightless rod ($0 < x < 1$) and of their velocities is not important from the point of view of further motion of the body G.

Let us now pass to solving the control problem (2.3)—(2.7). Integrating the first equation of (2.3) with the boundary conditions (2.4) and second relation of (2.3) both taken into account, yield the boundary value problem of the form (y is the deflection of the mass G from the OX axis)

$$I(x)u_{xx} = \varphi^{\cdot 2}u + x(M - \varphi^{\cdot 2}y) - M \tag{2.8}$$

$$y(t) \equiv u(t,\ 1),\ u(t,\ 0) = u_x(t,\ 0) \equiv 0$$

Let $W(\omega,\ x,\ s)$ be the Green's function for the equation $z'' = \omega^2 I^{-1} z$ where ω is a parameter. Then the function sought is

$$u(t,x) = \int_0^x W(\varphi^{\cdot},x,s)\,[(M - \varphi^{\cdot 2}y)\,s - M]\,\frac{ds}{I(s)} \tag{2.9}$$

We note that the function W can be constructed using a particular solution of the homogeneous equation (2.8). To construct a numerical solution of the problem (2.8) it is convenient to reduce it to the integral Volterra equation of second kind (t is a parameter)

$$u(t,x) = \varphi^{\cdot 2}\int_0^x \frac{x-s}{I(s)}\,u(t,s)\,ds + \int_0^x \frac{x-s}{I(s)}\,[(M - \varphi^{\cdot 2}y)\,s - M]\,ds \tag{2.10}$$

The well known method of consecutive approximations in extremely powerful when the initial approximation is well chosen. In one particular, important from the practical point of view case of $I = I_0 = \mathrm{const}$, we have

$$W(\varphi^{\cdot},x,s) = (\sqrt{I_0}/\varphi^{\cdot})\,\mathrm{sh}\,[(\varphi^{\cdot}/\sqrt{I_0})(x-s)]$$

Therefore for the functions $I(x)$ which are nearly constant, the initial approximation can be taken in the form (2.9) for which W is given above. Further, if the approximation $I(x) \simeq I_0 (1 - \beta x)^2$ where $1 - \beta > 0,\ \beta = \mathrm{const}$ is correct, then the Euler type equation (2.8) can be reduced by choosing the independent variable x thus: $1 - \beta x = e^{\xi}$ to a linear equation with constant coefficients for which the Green's function is constructed exactly as before. Assuming that the conditions of motion are such that the quantity $\varphi^{\cdot 2}$ is small, we obtain an approximate expression for the Green's function $W \simeq (x - s)\,/\,I(s)$ which can be utilized in the recurrent scheme of the method of consecutive approximations.

Substituting the known function $u(t,x)$ computed from (2.9) or (2.10) into the right-hand side of (2.8) we obtain, in accordance with (1.5), the expression for the moment of elastic forces.

Let us find the elastic deflection $y(t)$ of the body G using (2.9) with $x = 1$

$$y(t) = -M(t)\Omega^{-2}(\varphi^{\cdot}),\ \Omega^2 = b(\varphi^{\cdot}) / a(\varphi^{\cdot}),\ a > 0,\ b \geqslant 1 \tag{2.11}$$

$$a(\varphi^{\cdot}) \equiv \int_0^1 W(\varphi^{\cdot}, 1, x)\frac{1-x}{I(x)}\,dx,$$

$$b(\varphi^{\cdot}) \equiv 1 + \varphi^{\cdot 2}\int_0^1 W(\varphi^{\cdot}, 1, x)\frac{x}{I(x)}\,dx$$

From (2.11) it follows that the deflection $y = 0$ when $M = 0$. Integrating the second equation of (2.3) with respect to t and taking (2.5) into account, we have

$$\varphi^{\cdot} + y^{\cdot} - \varphi_0^{\cdot} - g(1) = \int_0^t M(\tau)\,d\tau \tag{2.12}$$

$$\varphi + y - [\varphi_0^{\cdot} + g(1)]\,t - \varphi_0 - f(1) = \int_0^t (t-\tau)M(\tau)\,d\tau$$

The relations (2.11), (2.12) obtained yield the solutions of the control problems (2.3)— (2.7) in the quasistatic approximation. Thus for the problem of rotation of elastic system as a whole with the final condition (2.7), the function $M \in K$ twice differentiable in t is such, that

$$M(0) = -f(1)\,\Omega^2(\varphi_0^{\cdot}),\ M(T) = 0 \tag{2.13}$$
$$M^{\cdot}(0) = -g(1)\,\Omega^2(\varphi_0^{\cdot}) - f(1)\,\Omega^{2\prime}(\varphi_0^{\cdot})\,\varphi^{\cdot\cdot}(0),\ M^{\cdot}(T) = 0$$

is chosen from the condition

$$\varphi_*^{\cdot} - \varphi_0^{\cdot} - g(1) = \int_0^T M(t)\,dt \tag{2.14}$$

Similarly, for the problem of rotating into a specified angular position with the final condition (2.6), the following relations must hold in addition to (2.13):

$$-\varphi_0^{\cdot} - g(1) = \int_0^T M(t)\,dt,\quad \varphi_* - [\varphi_0^{\cdot} + g(1)]\,T - \varphi_0 - f(1) = \int_0^T (T-t)M(t)\,dt \tag{2.15}$$

We note that the initial conditions of particular type (2.13) imposed on the control function $M(t)$ are stipulated by the degeneracy of the problem at $\varepsilon = 0$.

Let us consider a particular case, important from the practical point of view, of the zero initial conditions for the elastic deflections and velocities $f(x) = g(x) \equiv 0$ (see (1.9) and (2.5)). Then from (2.9), (2.11) and (2.12) it follows that the relations $M(0) = M^{\cdot}(0) = 0$ must hold. A typical form of the control functions $M(t) \in K$ is shown in Fig.2 for conditions (2.14) (curve 1) and (2.15) (curve 2). We note that in the case of a controlled rotational motion of a loaded rod the zero initial conditions are the most natural choice in the qusistatic approximation. This simplification of the problem is caused by the fact that when we put $M \equiv 0$ at the initial interval, then a rod of low mass will arrive at the state of equilibrium very rapidly (compared with the period $2\pi / \Omega$) of the quasistatic oscillations), in the presence of dissipation which always occurs in practice.

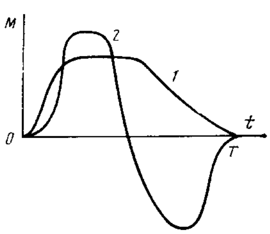

Fig.2

Let us now consider a relative motion of the body G in the case when the motion of the tangent to OX is specified kinematically, i.e. $\varphi^{\cdot\cdot}$ is a given function of $\gamma(t)$ and γ is regarded as a control. Then the deflection $y(t)$ will, in accordance with (2.3) and (2.11), be given by

$$y^{\cdot\cdot} + \Omega^2(\varphi^{\cdot})y = -\gamma(t),\ y(0) = f(1),\ y^{\cdot}(0) = g(1) \tag{2.16}$$
$$\varphi^{\cdot}(t) = \varphi_0^{\cdot} + \int_0^t \gamma(\tau)\,d\tau,\quad \varphi(t) = \varphi_0 + \varphi_0^{\cdot}t + \int_0^t (t-\tau)\gamma(\tau)\,d\tau$$

The quantity Ω^2 in (2.16) characterizes the oscillation frequency of the mass G. In particular, for $\gamma \equiv 0$, $\Omega = \text{const}$ we have the equation of a linear oscillator. For a clamped rod ($\varphi^{\cdot} \equiv 0$) the value of $\Omega(0)$ yields the frequency of the quasistatic oscillation of a centilever wfth mass

m at the end. When in particular case the rod is homogeneous, the function $\Omega^2(\varphi^\cdot)$ is equal to /3/

$$\Omega^2(\varphi^\cdot) = \frac{\alpha^3 \operatorname{sh}\alpha}{3(\alpha \operatorname{ch}\alpha - \operatorname{sh}\alpha)}\Omega^2(0), \qquad \alpha = |\varphi^\cdot| I_0^{-1/2}, \qquad \Omega^2(0) = 3I_0$$

We note that when $\alpha \sim 1$, the coefficient accompanying $\Omega^2(0)$ is very nearly equal to unity $1 + \alpha^2/15 + \ldots$. Thus the strengthening effect becomes appreciable at high velocities $\varphi^\cdot$: for $\alpha^2 \sim 10$.

When the system (2.16) is essentially nonlinear, solution of the problem of control of the motion, including the control optimal with respect to some criteria, presents considerable difficulties. In case when we can assume that $\Omega^2 \approx \Omega^2(0)$, we obtain a control problem equivalent to the problem of displacement or acceleration of an oscillator /4,10/.

Having found the functions $\gamma(t)$ and $y(t)$, we substitute them into the second equation of (2.3). This yields the magnitude of the control moment $M(t)$ needed for the realization of the motion in question. We can obtaine the solution of the problem of control at $\varepsilon \neq 0$ with any prescribed degree of accuracy in ε, using the methods of the perturbation theory. In this case we must assume a sufficiently high degree of smoothness in the control function $M(t)$, $t \in [0, T]$.

2°. Next we consider the control problem (1.6)—(1.11) in the other limiting case when the rigidity of the rod is very high, while the amplitude and period of the natural oscillations are essentially small. In this case it is convenient, when passing to the dimensionless coordinates in accordance with the formulas (2.1), to take $\nu = (M_0 / J)^{1/2}$ where M_0 is a characteristic quantity with dimension of the moment of forces, e.g. $M_0 = \sup_t |M(t)|$, J is the characteristic quantity with dimension of the moment of inertia of the rod with mass G. Then the quantity ν^2 will characterize the angular acceleration of rotation of the system as a whole. In the new variables the equations of motion (1.6), (1.8) will become (with the primes omitted)

$$\mu\varkappa\rho(x)u_{tt} + [I(x)u_{xx}]_{xx} = \mu\varkappa\varphi^{\cdot 2}\left[u_{xx}\int_x^1 \rho(s)s\,ds - \rho(x)xu_x + \rho(x)u\right] + \mu\chi u_{xx} - \mu\varkappa\rho(x)x\varphi^{\cdot\cdot} \tag{2.17}$$

$$\varphi^{\cdot\cdot} + \varkappa\int_0^1 \rho(x)xu_{tt}(t,x)\,dx + \chi u_{tt}(t,1) = K(t), \qquad \mu = \frac{M_0 l}{EI_0}, \quad \varkappa = \frac{\rho_0 l^3}{J}, \quad \chi = \frac{ml^2}{J}, \quad K = \frac{M}{M_0} \tag{2.18}$$

The boundary and initial conditions will now become

$$u(t,0) = u_x(t,0) = u_{xx}(t,1) = 0 \tag{2.19}$$

$$[I(x)u_{xx}]_x|_{x=1} = \mu\chi(u_{tt} + \varphi^{\cdot\cdot} - \varphi^{\cdot 2}u + \varphi^{\cdot 2}u_x)_{x=1}$$

$$u(0,x) = \mu f(x),\ u_t(0,x) = \mu g(x)$$

Depending on the formulation of the problem of control, the conditions at the completion of the process will have the form (1.10) or (1.11). We note that in the formulation used here f, g and K must satisfy certain additional requirements (see below).

We construct the solution using the method of perturbation theory, in the powers of small parameter μ, assuming the function $K(t)$ to be sufficiently smooth. In the limit (when $\mu = 0$) we have $u^c(t,x) \equiv 0$, $(\varphi^c)^{\cdot\cdot} = M^c(t)$. The function $M^\circ(t)$ is obtained from a relation of the type (2.14) or (2.15) taken as $g = f \equiv 0$. Function φ^c is also found. Setting

$$u = \mu u^1(t,x) + \mu^2 \ldots, \quad \varphi = \varphi^c(t) + \mu\varphi^1(t) + \mu^2 \ldots$$

$$K = M^c(t) + \mu M^1(t) + \mu^2 \ldots$$

and using (2.19) we obtain the following relations for the unknown functions u^1, φ^1, M^1:

$$[I(x)u_{xx}^1]_{xx} = -\varkappa M^\circ(t)x\rho(x) \tag{2.20}$$

$$(\varphi^1)^{\cdot\cdot} + \varkappa\int_0^1 \rho(x)xu_{tt}^1(t,x)\,dx + \chi u_{tt}^1(t,1) = M^1(t)$$

$$u^1(t,0) = u_x^1(t,0) = u_{xx}^1(t,1) \equiv 0, \quad [I(x)u_{xx}^1]_x|_{x=1} = \chi M^c(t)$$

$$u^1(0,x) = f(x), \quad u_t^1(0,x) = g(x)$$

Solving the boundary value problem (2.20) for u^1, we obtain

$$u^1(t,x) = -M^\circ(t)\int_0^x \frac{x-s}{I(s)}n(s)\,ds$$

$$n(x) \equiv \varkappa \int_0^x (x-s) s\rho(s)\,ds + \chi(1-x) - \varkappa_* x + \varkappa_{**}$$

$$\varkappa_* = \varkappa \int_0^1 x\rho(x)\,dx, \qquad \varkappa_{**} = \varkappa \int_0^1 x^2\rho(x)\,dx$$

In this manner we obtain the function $u^1(t, x)$ using the zero approximation to the control $M^\circ(t)$ corresponding to the problem of control of a perfectly rigid rod. Substituting u^1 into the second equation of (2.20) we obtain an exceptionally simple relation for determining the unknown φ^1 and M^1 of the form $(\varphi^1)^{\cdot\cdot} + c\,M^{\circ\cdot\cdot}(t) = M^1$ where is a known constant. By virtue of the choice of φ°, the functions φ^1 and M^1 can be arbitrary, but such that the initial and final conditions are satisfied, e.g. $M^1(t) \equiv cM^{\circ\cdot\cdot}(t)$, $\varphi^1 \equiv 0$. We note that the initial distribution in (2.20) can be satisfied only when the following relation holds:

$$f(x) \equiv -M^\circ(0) \int_0^x \frac{x-s}{I(s)} n(s)\,ds, \quad g(x) \equiv -M^{\circ\cdot}(0) \int_0^x \frac{x-s}{I(s)} n(s)\,ds$$

In particular, if $f(x) = g(x) \equiv 0$,, then the initial conditions hold when $M(0) = M^\cdot(0) = 0$. For $t \geqslant T$ we must also put $M(t) = M^\cdot(t) \equiv 0$. Thus the control problem with terms of the order of μ taken into account, is solved.

The smallness of the parameter μ in (2.17) and (2.18) can be treated differently, namely, assuming the period of the natural oscillations to be of the order of unity, we assume that the angular acceleration is low (time of rotation is long). Such an approach does not demand the high degree of smoothness of the control function $M(t)$. However, in this case it is necessary to solve the control problem with damping of an enumerable number of oscillation modes taken into account /5—8/. The corresponding algorithm of the approximate solution can be constructed in the form of an expansion in the powers of

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